Report on “The Thermodynamic Meaning of Relative Entropy”

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1 Introduction

Thermodynamics and statistical mechanics are, at their core, observer dependent theories of statistical inference. Given the nature of one’s beliefs, they provide a mechanism for computing the likelihood of possible outcomes of observations consistent with those beliefs in such a way as to minimize assumptions about missing information. Examples of this are the second law of thermodynamics and the Boltzmann distribution.

Because of this, many “physical” thermodynamic quantities are, in fact, subjective and observer dependent. The difference between heat and work, the concept of reversibility, and the ideas of temperature and chemical potential are observer dependent. For example, the quantity of “heat” which passes from one system to another is nothing more than the energy which passes between those systems by way of physical degrees of freedom for which an observer does not (or chooses not to) have information–the concepts of macroscopic and microscopic are not fundamental to thermodynamics. However, it should be mentioned that all observers, regardless of information, will conclude that the net entropy should be non-decreasing in an isolated system.

In this report we will focus on thermodynamic work; in particular, we will show that the amount of work which an observer must provide to reset a bit is dependent on the knowledge which that observer has about the bit. We will see that the existence of quantum correlations (which are in some sense “stronger” than classical correlations) can provide an observer with more knowledge about the state of a bit than would be allowed by even the strongest classical correlation. As a generalization of Landauer’s principle, this will turn out to provide a work *yield* in a situation where any classical correlations could provide none.

In what follows, we will consider a physical process which can reset a bit given quantum information while providing a net yield in work. We will write a general formula for the work yield in such reset processes which can be considered a generalization of Landauer’s principle (generalized only in the sense that it allows for varying amounts of information about the system being erased). In the case of a quantum reset process we will see that the second law of thermodynamics is upheld by destroying one unit of entanglement in the process. We will also learn that, in the same way, there is a thermodynamic significance to quantum discord, namely, that discord represents the gain in work yield achievable in principle between a classical observer and a quantum one.

2 Generalized Landauer's Principle

Nowadays, Landauer's principle is generally stated as follows: "A physical process which performs a logically irreversible operation must cause an increase in thermodynamic entropy in the environment or in the body of the apparatus which performs it." This may be the only statement of the principle which could be considered universally accepted. In order to create the increase in entropy of the environment it has been reported that one must pay a "price" in some conserved macroscopic quantity in the system, such as angular momentum or energy. Typically, it is energy in the form work which is provided. Then, one could say: "A physical process which performs a logically irreversible operation must require an input of work which is converted to heat in the apparatus or environment." In the case of the erasure of a bit of information this statement now reduces to: "A physical process which erases a bit requires a minimum amount of work $kT\ln(2)$ to be provided and which will be converted to heat in the environment." This is essentially the way that Landauer originally stated it. Some have attempted to go further and state: "A physical process which erases a bit requires a minimum amount of work $kT\ln(2)$ and is thermodynamically irreversible." Irreversibility in this case is observer-dependent; an observer who knows the initial physical state of the bit will conclude irreversibility, while one which is ignorant of the initial state will conclude reversibility. However, since, from the point of view of an observer who is ignorant of the initial state, the logical process performed is no longer logically irreversible, then Landauer's principle doesn't apply to begin with. Therefore, it can confidently be said that, for an isolated system to truly erase a bit, work $kT\ln(2)$ must be provided and dissipated as heat in the environment, and that the process will be thermodynamically irreversible.

However, here we're more interested in the work requirement than the question of reversibility. The essential reason for the required work input in Landauer's principle is that the physical apparatus performing the erasure (which, by assumption, cannot have any information about the bit it is erasing) must perform the erasure using a process which is not conditioned on the state of the bit. In other words, the computer must have one single "bit erasing" physical operation which is performed on all bits when erasing, regardless of their physical state. Now, the fundamental fact that classical and quantum dynamics of isolated systems are information-preserving leads one to conclude that this process would be impossible in an isolated computer unless the number of microstates associated to each logical (1, 0) state were increased by a factor of two. This can only be done by introducing energy into the non-information-bearing degrees-of-freedom of the computing apparatus. In other words, we must perform work on the system which will ultimately be converted to "heat."

However, one can imagine that, if an observer were in fact able to condition the reset process on the physical state of the bit, then it could be done with no work input, since either the "bit-flip" or the "identity" operation would be separately logically reversible. So, if one has complete information about the bit it can be erased with zero work. If one has no information about the bit, or proceeds as if that were the case (as in Landauer's principle above), it costs $kT\ln(2)$. This suggests the formula $w = S(X|O)kT$, where w is the work required, $S(X|O)$ is the information entropy of X (the bit) conditioned on a memory O, and kT is Boltzmann's constant multiplied by the temperature of the environment or computing device.

Now, a fundamental requirement for an erasure process is that we not disturb the memory in the process, lest we render our theoretical results invalid (i.e., the memory could then take entropy from the environment, removing the need to apply work). Therefore, we must have $\Delta S(O) = 0$, where Δ signifies the change resulting from the erasure process. In fact, the total entropy change of such a process is

$$\begin{aligned}\Delta S(XOT) &= \Delta S(XO) + \Delta S(T) - \Delta S(XO:T) \\ &= \Delta S(X|O) + \Delta S(O) + \Delta S(T) \\ &= \Delta S(X|O) + \frac{w}{kT} \\ &\geq 0.\end{aligned} \tag{1}$$

where we've discarded the mutual information between the heat reservoir (T) and bits (XO) due to the nature of typical thermodynamic observers, we've let $\Delta S(O) = 0$, have written the entropy change of the reservoir in terms of the work w which we perform, and have enforced the second law of thermodynamics. This implies that $w \geq -\Delta S(X|O)kT$. Since we are performing a reset procedure, all observers know that the final state of X is zero. Therefore $\Delta S(X|O) = 0 - S(X|O)_i$, where the subscript denotes the initial state. Henceforth we will only refer to the entropy $S(X|O)_i$ of the initial state, and so we will drop the subscript. We now have

$$w \geq S(X|O)kT. \tag{2}$$

Is this lower bound imposed by the second law of thermodynamics achievable? The answer is yes, and in the next section we give an explicit example.

2.1 A Thermal Process for Erasing an Unknown Bit

Consider a bit for which we have no information. The state we assign is a completely mixed state with equal probabilities for zero and one. Also consider that the Hamiltonian of the system which is holding the bit is completely degenerate, so energy of the bit is $E_0 \equiv 0$. Now if we put this bit into contact with a heat reservoir at temperature T it will be randomized (however without exchanging any energy or increasing total entropy). Now, let's say that we slowly alter the Hamiltonian of the bit's system so that the energy of the one state is taken up to $+\infty$. By integrating over the Boltzmann distribution we can see that this takes an amount of work equal to $kT\ln(2)$:

$$\begin{aligned} w &= \int_0^\infty \bar{n}(E)\; dE = \int_0^\infty \frac{e^{-E/kT}\; dE}{1+e^{-E/kT}} \\ &= kT\ln(2), \end{aligned} \tag{3}$$

where $\bar{n}(E)$ is the occupation statistics of an energy level E given by the Boltzmann distribution. Essentially, the higher we push, the less likely the bit is to be in the one state, so pushing to ∞ costs only a finite amount of work. Since the one state is at an extremely high energy, the state of the bit is almost surely in the zero state. Now we decouple the bit front the reservoir, lower the energy level of the one state back to zero, and hence the bit has been erased at the expense of work $kT\ln(2)$. This process is both logically and thermodynamically reversible given that we had no knowledge of the state of the bit to be erased. Note that when we lower the energy level back to zero at the end we do not get the work back since the bit is no longer in contact with the heat reservoir and so is never present in the one state as we lower it. In this example we have reached the lower bound of Eq. 2, given that our initial information was $S(X|O) = \ln(2)$–maximal uncertainty. A similar process con be constructed to thermally erase a bit in a known state, with a work cost of zero.

3 Negative Conditional Entropy

In quantum mechanics, the quantum conditional entropy is defined as $S(X|O) \equiv S(XO) - S(O)$. If X and O represent a pair of maximally entangled qubits, then $S(X|O) < 0$, something which is not possible for any state of classical correlation between X and O. We may ask, does Eq. 2 hold even in the case that the bits X and O are quantum mechanical and hence may be entangled? If so, does a value of $w < 0$ actually imply that we can *yield* work by resetting the bit? In fact, the answer is yes. In the next section we will see an explicit example of this process.

But before doing that, let's calculate the difference in work yield for a given state XO for an observer who can potentially take advantage of quantum correlations and one who cannot:

$$w_c - w_q = S(X|O_c)kT - S(X|O_q)kT = D(X|O)kT, \tag{4}$$

where $D(X|O)$ is the quantum discord w_c and w_q are the classical observer and quantum observer work costs, respectively, and where $S(X|O_c)$ is the entropy calculated by an observer who is only allowed to take advantage of classical correlations in a state. We see that the difference in work cost is related to the quantum discord–thereby giving that quantity a thermodynamic significance. Note that the maximally entangled state minimizes Eq. 2 and maximizes discord. Therefore, if we can find a reversible process which erases a bit given a maximally entangled memory bit, we will yield $kT\ln(2)$ of work and will have performed as well as nature allows with regard to work extraction.

4 A Simple Work Extraction Process for Bit Erasure With Quantum Memory

To extract $kT\ln(2)$ of work from an entangled bit pair XO, first perform a nonlocal rotation (which is not forbidden) on the bits to place them into the $|0\rangle \otimes |0\rangle$ state. Already we have reset the bit X to zero! However, we cannot be finished, for we have altered the reduce state of the memory bit O from a completely mixed state to a pure state. In order to restore it we need only place it in contact with a heat reservoir. However, in doing this, we can take advantage of the 'reverse erasure' procedure, whereby one can extract $kT\ln(2)$ of work by allowing a bit to be randomized. This process is nothing but the reverse of the thermal erasure process discussed earlier. After performing this the bit O has been returned to a fully mixed state and we have received the work $kT\ln(2)$ which we predicted we could obtain. One may wonder why it is necessary to 'remix' the bit O since, is it not the case that a $|0\rangle$ state is a possible physical instance of a completely mixed state? The answer is yes; however, had we not thermally mixed the bit we would have had no means of extracting work from the heat reservoir.

4.1 Is the Second Law Upheld?

In the process just discussed we have erased a bit whose reduced state was initially completely mixed, have left the memory bit's reduced state unchanged, have extracted work from a heat reservoir, and have left the environment otherwise unchanged. One may wonder why the second law of thermodynamics doesn't appear to have been broken in this situation. The reason is that the entanglement between the bits was broken in the process. If we consider the observer who knows the initial joint state of the bits then the initial entropy is zero. After the process, this observer perceives that the entropy of the reservoir has decreased but that the entropy of the joint state of the bits has increased by that same amount. Hence the process is reversible. However, we cannot simply repeat this procedure, continually "erasing" the bit and extracting work. In fact, the joint state of the bits is now mixed, so it cannot be restored to a pure (entangled) state without performing work and, consequently, losing what we have gained. Therefore, we cannot operate this process on a cycle without introducing new entangled bit pairs.

5 Conclusions

In conclusion, we have derived a lower bound on the work required to erase a bit in the general case where the information can be classical or quantum. This generalized Landauer's principle supports a lower bound which can be negative, meaning that a reset process could, in principle, yield work in the case of quantum information. We have shown (by explicit example) that this lower bound is achievable and that it does not violate the second law of thermodynamics from the point of view of the observer performing the reset.

In doing this it has become clear that the concepts of negative entropy, entanglement and quantum discord have physical significance. Their presence (provided an observer is aware of it) can increase the net amount of work which can be extracted during a given thermal process. This is because the presence of these low-entropy states permits the observer to extract an extra unit of work (by destroying them) than would have been possible in the presence of only classical correlations.

References

[1] L. del Rio et al, "The thermodynamic meaning of negative entropy," Nature 474, 61 (2011)